

IT formulae for gamma target: mutual information and relative entropy

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Abstract

In this paper, we introduce new Stein identities for gamma target distribution as well as a new non-linear channel specifically designed for gamma inputs. From these two ingredients, we derive an explicit and simple formula for the derivative of the input-output mutual information of this non-linear channel with respect to the channel quality parameter. This relation is reminiscent of the well-known link between the derivative of the input-output mutual information of additive Gaussian noise channel with respect to the signal-to-noise ratio and the minimum mean-square error. The proof relies on a rescaled version of De Bruijn identity for gamma target distribution together with a stochastic representation for the gamma specific Fisher information. Finally, we are able to derive precise bounds and asymptotics for the input-output mutual information of the non-linear channel with gamma inputs.

Key words: non-linear channel, mutual information, relative entropy, Fisher information, estimation theory.

1 Introduction

Let X, Y be two random variables on the same probability space, with joint probability measure $P^{X,Y}$ and marginals P^X and P^Y , respectively. We choose the law of the couple (X, Y) to be absolutely continuous with respect to a common dominating measure μ and denote $p^{X,Y}(x, y)$, $p^X(x)$ and $p^Y(y)$ the corresponding Radon-Nikodym derivatives. The *mutual information* between X and Y is

$$I(X; Y) = E \left[\log \left(\frac{p^{X,Y}(X, Y)}{p^X(X)p^Y(Y)} \right) \right]. \quad (1)$$

Mutual information satisfies $I(X; Y) \geq 0$ with equality if and only if X and Y are independent and therefore mutual information captures the dependence between X and Y . The *relative entropy* (a.k.a. Kullback-Leibler divergence) from Y to X is

$$D(X||Y) = E [\log(p^X(X)/p^Y(X))]. \quad (2)$$

Relative entropy satisfies $D(X||Y) \geq 0$ with equality if and only if $X \stackrel{\mathcal{L}}{=} Y$ and therefore $D(X||Y)$ captures the difference between $\mathcal{L}(X)$ and $\mathcal{L}(Y)$. One speaks of Gaussian relative entropy if Y is standard Gaussian.

Mutual information and relative entropy are crucial in a wide variety of fields (see e.g. [30] for an overview) but are both generally analytically, and even in some cases algorithmically, intractable.

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It is thus useful to dispose of formulas allowing to control them in terms of quantities which are more amenable to computations. Two such formulas are Stam’s *De Bruijn identity* [26, 7] and Guo, Shamai and Verdú’s *MMSE identity* [10] (which we shall refer to as GSV identity in the sequel). Exact statements of these identities are deferred to Section 2. Informally, the De Bruijn identity provides an explicit link between the Gaussian relative entropy of an absolutely continuous random variable X and the *Fisher information* of X . Similarly, the GSV identity provides an explicit link between the mutual information in a Gaussian channel and the *minimal mean square error* in said channel. Both formulas are, as it turns out, essentially equivalent because either can be deduced - at least formally - from the other, see Sections 2 and 6. They relate information theoretic quantities (relative entropy, mutual information) to quantities typically of interest in statistical estimation theory (Fisher information, MMSE) and have proven to be linchpins of important developments in contemporary information theoretic probability theory (e.g. for entropic CLTs [6, 14, 15, 3, 2, 29], analysis of additive Gaussian channels [18, 10] or, more generally, IT inequalities [12, 25]).

These two equivalent identities are inherently of a Gaussian nature and it is therefore natural to enquire whether similar relationships also hold outside of the Gaussian realm. Quoting [11], “a natural question to pose is how general the information-estimation relationship can be”. This important question has of course already received a lot of attention in the literature and there exist De Bruijn identities, on the one hand, and GSV identities, on the other hand, for most classical target probability distributions of practical relevance (precise references will be given later in the text). The resulting identities, however, no longer enjoy the elegance and ease of manipulation as their Gaussian counterparts. In particular the estimation quantities derived in either cases do not bear natural interpretations and, to the best of our knowledge, the equivalence between the general-target De Bruijn and the general-target GSV identities has never been investigated. As is evident from an inspection of their proofs, both the De Bruijn and the GSV identities are obtained through a study of entropy/information jumps around X along small perturbations of the form

$$X \mapsto X_r := \sqrt{r}X + N \quad (3)$$

with $r > 0$ and N an independent standard Gaussian. A first intuitive way to branch outside of the Gaussian scope is to work as in [11, 23] and extend (3) by considering more general (additive or even non-additive) noising mechanisms of the form $X \mapsto X_a = h(a; X; W)$ where a is a real parameter, $h(\cdot; \cdot; \cdot)$ is a deterministic function and W is an independent noise following some arbitrary distribution. As could be expected, the classical MMSE quantity from estimation theory no longer plays any role in these identities, and the corresponding correct object is expressed as the correlation of two generally intractable conditional expectations (depending on log-derivatives of the density of X_r) which bears no explicit representation nor interpretation.

Now, depending on the context, the perturbation X_r in (3) is referred to as an “additive Gaussian channel” [30], a “smart path” [20] or an “Ornstein-Uhlenbeck evolute” around X [5]. The key fact here is that the deformation $x \mapsto \sqrt{r}x + N$ arises naturally through the action of the Ornstein-Uhlenbeck semigroup and thus X_r ought to be interpreted as a stochastic representation for the “smart path” interpolation between the law of X and the Gaussian distribution. A general take on this semi-group interpretation leads to De Bruijn-type formulas with general reference probability measure (see [4, 5]) providing a direct link between the relative entropy $D(X || Z)$ from a target random variable Z to a random variable X and a *target-specific* Fisher information structure. This Fisher information structure is, in general, implicit as it depends on the distribution of the *ad hoc* deformation X_r which bears no explicit stochastic representation equivalent to (3).

In this paper we derive a new set of De Bruijn/GSV identities specifically when the target distribution is in the family of gamma distributions (which encompass as particular cases the chi-square and exponential distributions). There are two main new ingredients behind our results. The first ingredient is a family of *Stein identities* for gamma target distribution. Stein identities are characterizations of probability distributions through the action of target-specific differential operators (see e.g. the Gaussian Stein identity (4)). They are available for virtually any probability distribution allowing a closed form distribution ([16]) and are known to lie at the boundary between IT and estimation

theory [8, 24, 17, 22, 21]. The second ingredient is a new noising-channel $r \mapsto X_r$ specifically designed for gamma input (see (35)); interestingly this channel is quadratic rather than linear as in (3). By combining these two concepts we derive, via elementary arguments, tractable and interpretable gamma-specific De Bruijn and GSV identities. While the De Bruijn identity is in essence a rescaling of known results from our previous paper [1], the gamma-GSV identity we obtain is entirely new. We prove that our quadratic channel has properties which are strikingly similar to the additive Gaussian channel in terms of mutual information and its asymptotics for large values of the channel quality parameter.

1.1 Outline of the paper

In Section 2 we review the relevant known results for Gaussian target. In Section 3 we provide the necessary IT and Stein identities for gamma target and we also recall the *ad hoc* gamma-specific De Bruijn identity (Theorem 2). In Section 4 we discuss the main properties of the gamma-counterpart to the smart path (3) and in Section 5 (mainly Proposition 5) we provide the key ingredient of the paper, namely a new representation of the (gamma-specific) Fisher information in terms of a quantity reminiscent of the minimal mean square error at the heart of the GSV equality. In Section 6 we show that the quantities we have introduced are indeed the missing link between IT and estimation theory with gamma target: we derive an explicit GSV formula for gamma target as well as fine upper bounds for the variation of the mutual information with respect to the channel quality parameter. The bounds are universal to the extent that they depend on the distribution of the input only through its mean and the estimation theoretical quantity put forward in Proposition 5. The only assumption needed on the input X is the existence of finite $\alpha + 4$ moment. Finally, for gamma input with parameters (α, λ) , we obtain an inequality on the mutual information reminiscent of the Gaussian case and for $\alpha = 1/2$, we obtain the exact asymptotic for large values of the channel quality parameter of the input-output mutual information.

2 IT and Stein Identities for Gaussian target

Let N be a standard Gaussian random variable with pdf $\gamma(x) = (2\pi)^{-1/2}e^{-x^2/2}$. Stein's well-known identity [28, 27] states that:

$$E[N\phi(N)] = E[\phi'(N)] \text{ for all } \phi \in \mathcal{F}(N) \quad (4)$$

with $\mathcal{F}(N)$ the collection of absolutely continuous test functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\phi' \in L^1(N)$. Moreover if another random variable X also satisfies (4) then $X \stackrel{\mathcal{L}}{=} Z$. We refer the reader to [20, Lemma 3.1.2] for a streamlined proof. Extending identity (4) to arbitrary target entices us to associate to any random variable X with mean μ and variance σ^2 a random variable $\rho_X(X)$ defined (almost everywhere) through the identity:

$$E[\rho_X(X)\phi(X)] = -E[\phi'(X)] \text{ for all } \phi \in \mathcal{F}(X) \quad (5)$$

with $\mathcal{F}(X)$ the collection of absolutely continuous test functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\phi' \in L^1(X)$. The random variable $\rho_X(X)$ defined a.e. by (5) is called the score of X ; it is easy to see that if X has differentiable density p_X which cancels at the border of its support then $\rho_X(X) = \frac{d}{dx} \log p_X(x)|_{x=X}$ satisfies (5). In particular from (4) we know that $\rho_X(X) = -\frac{X-\mu}{\sigma^2}$ if and only if $X \stackrel{\mathcal{L}}{=} \sigma N + \mu$ (here and throughout we reserve the notation N for a standard normal random variable). Conditions on the distribution of X under which the score is well-defined have been thoroughly addressed in the literature and it is a well-known fact that the score is essentially unique in the sense that if a random variable Y satisfies (5) with the same score as X then $Y \stackrel{\mathcal{L}}{=} X$; see [27, 14, 16]

Applying (5) to the test function $\phi(x) = 1$ we deduce that if X admits a score then necessarily $E[\rho_X(X)] = 0$. The second moment of the score plays a role in the *standardized relative Fisher*

information

$$J_{\text{st}}(X) = \sigma^2 E [(\rho_X(X) + (X - \mu)/\sigma^2)^2] = \sigma^2 E [\rho_X(X)^2] - 1 \quad (6)$$

and it is well known that $J_{\text{st}}(X) = 0$ if and only if $X = \sigma N + \mu$ (see e.g. [14, 15, 22]); in other words the second moment of the score suffices to characterize the distribution. The quantity $I(X) = E[\rho_X(X)^2]$ is the *Fisher information* of X and, from previous considerations we know that $I(\sigma N + \mu) = \frac{1}{\sigma^2}$.

Relative entropy (2) and standardized Fisher information (6) are related through the classical De Bruijn identity

$$\frac{d}{dr} D(X_r || N) = \frac{1}{2r} \left(I(X_r) - 1 + r \right), \quad (7)$$

$$= \frac{1}{2(1+r)} \left(r + \frac{1}{r} J_{\text{st}}(X_r) \right), \quad (8)$$

still with X_r as in (3) (see [7, 14] for a proof of (8) solely under moment assumptions on X). Applying a conditional version of (4), we note how for all sufficiently regular test functions ϕ we also have

$$E [(E[\sqrt{r}X | X_r] - X_r)\phi(X_r)] = -E [N\phi(X_r)] = -E [\phi'(X_r)]$$

from which we deduce the representation

$$\rho_r(X_r) = \sqrt{r} E[X | X_r] - X_r \quad (9)$$

for $\rho_r(X_r)$ the score of X_r . This in turn leads to the representation of standardized Fisher information:

$$\frac{1}{r} J_{\text{st}}(X_r) = 1 - (1+r) E [(X - E[X | X_r])^2] \quad (10)$$

which provides a connection between Fisher information (and hence relative entropy) with estimation theory's Minimal Mean Square Error

$$\text{MMSE}(X, Y) = E [(X - E[X | Y])^2]. \quad (11)$$

Plugging (10) into (7) we obtain

$$\frac{d}{dr} D(\sqrt{r}X + N || N) = \frac{1}{2} (1 - \text{MMSE}(X, X_r)), \quad (12)$$

which is equivalent to formula (66) of Theorem 5 page 7 of [10].

Let X be centered with finite variance. As already touched upon in the introduction, the original GSV formula from [10] links mutual information (1) and MMSE (11) through:

$$\frac{d}{dr} I(X; X_r) = \frac{1}{2} \text{MMSE}(X, X_r). \quad (13)$$

We conclude this section by showing how to obtain (13) from (12); our argument relies on ideas from Section II-D of [10]. We stress that our method of proof is robust towards a change of channel, in the sense that we will show in Section 6 how it can be transposed from the Gaussian to the gamma setting. For $r > 0$ we first set $\tau(r) = r/(r+1)$ and introduce the random variables

$$\begin{aligned} \tilde{X}_{\tau(r)}(x) &= \sqrt{\tau(r)}x + \sqrt{1-\tau(r)}N \\ X_r(x) &= \sqrt{r}x + N \end{aligned}$$

with $x \in (-\infty, +\infty)$ and N , as above, an independent standard Gaussian. We also set $X_r = X_r(X)$ and $\tilde{X}_{\tau(r)} = \tilde{X}_{\tau(r)}(X)$. For any deterministic functional, we denote by $E_X[F(\tilde{X}_{\tau(r)}(X))]$ (similarly with $X_r(X)$) the following type of integral:

$$E_X[F(\tilde{X}_{\tau(r)}(X))] = \int p_X(x) F(\tilde{X}_{\tau(r)}(x)) dx. \quad (14)$$

By standard arguments we know that $I(X, \tilde{X}_{\tau(r)}) = I(X, X_r)$ and

$$I(X, \tilde{X}_{\tau(r)}) = E_X \left[D(\tilde{X}_{\tau(r)}(X) \parallel N) \right] - D(\tilde{X}_{\tau(r)} \parallel N), \quad (15)$$

where $E_X[\cdot]$ denotes an expectation taken with respect to X . Using regularity arguments provided in [7] in combination with the chain rule we easily obtain:

$$\frac{d}{dr} \left(I(X, X_r) \right) = \frac{1}{r(r+1)} \left(E_X \left[J_{st}(\tilde{X}_{\tau(r)}(X)) \right] - J_{st}(\tilde{X}_{\tau(r)}) \right). \quad (16)$$

We can finally conclude.

Proposition 1. *[Theorem 1 in [10]] Identity (13) holds if X is centered with $E[X^2] = 1$.*

Proof. First note how for all x the random variable $\tilde{X}_{\tau(r)}(x)$ remains Gaussian so that straightforward computations lead to:

$$J_{st}(\tilde{X}_{\tau(r)}(x)) = \frac{1}{2} \frac{r(r+x^2)}{1+r},$$

for all real x . Next, by scaling arguments, we get

$$J_{st}(\tilde{X}_{\tau(r)}) = (1+r)J_{st}(X_r) - \frac{r^2}{2}. \quad (17)$$

Moreover, using (10) we obtain

$$J_{st}(\tilde{X}_{\tau(r)}) = \frac{1}{2} \left[(1+r)r(1 - \text{MMSE}(X, X_r)) - r^2 \right], \quad (18)$$

$$= \frac{1}{2} \left[-r(1+r) \text{MMSE}(X, X_r) + r \right] \quad (19)$$

Combining everything together, we have:

$$\begin{aligned} \frac{d}{dr} \left(I(X, X_r) \right) &= \frac{1}{2r(r+1)} \left(E \left[\frac{r(r+X^2)}{1+r} \right] + r(1+r) \text{MMSE}(X, X_r) - r \right), \\ &= \frac{1}{2} \text{MMSE}(X, X_r), \end{aligned}$$

as required. \square

3 IT and Stein identities for gamma target

Let Z be a gamma distributed random variable with pdf $\gamma_{\alpha, \lambda}(x) = \lambda^\alpha / \Gamma(\alpha) x^{\alpha-1} \exp(-\lambda x)$ over the positive half line. The equivalent of Stein's identity (4) for a gamma target has long been known to be

$$E[(\lambda Z - \alpha)\phi(Z)] = E[Z\phi'(Z)] \quad (20)$$

(see [19]). Moreover if some positive random variable X also satisfies this identity over an appropriately wide class of functions then $X \stackrel{\mathcal{L}}{=} Z$, see [9] for a proof. Introducing the derivative $\partial_x^\sigma \phi(x) = (\sqrt{x}\phi(x))'$, we rewrite (20) as

$$E \left[\sqrt{Z}(\lambda\sqrt{Z} - (\alpha - 1/2)/\sqrt{Z})\phi(Z) \right] = E \left[\sqrt{Z}\partial_x^\sigma \phi(Z) \right] \text{ for all } \phi \in \mathcal{F}(Z). \quad (21)$$

with $\mathcal{F}(Z)$ a collection of sufficiently smooth test function $\phi : \mathbb{R}_+^* \rightarrow \mathbb{R}$ such that $x \mapsto \sqrt{x}\partial_x^\sigma \phi(x) \in L^1(Z)$. While in appearance less elegant than (20), we claim that (21) is actually the correct starting point for Stein/IT analysis with a gamma target.

As in Section 2 we begin by extending the scope of (21) to arbitrary target by introducing for arbitrary positive X a random variable $\rho_X^\gamma(X)$ defined (almost everywhere) through the identity:

$$E \left[\sqrt{X} \rho_X^\gamma(X) \phi(X) \right] = -E \left[\sqrt{X} \partial_x^\sigma \phi(X) \right] \text{ for all } \phi \in \mathcal{F}^\sigma(X) \quad (22)$$

with $\mathcal{F}^\sigma(X)$ the collection of absolutely continuous test functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that $x \mapsto \sqrt{x} \partial_x^\sigma \phi(x) \in L^1(X)$. We call $\rho_X^\gamma(X)$ defined by (22) X 's γ -score. Taking $\phi(x) = 1/\sqrt{x}$ in (22) we conclude that if X admits a γ -score then necessarily it satisfies $E[\rho_X^\gamma(X)] = 0$. From (21) we know that the $\gamma(\alpha, \lambda)$ distribution is characterized by

$$\rho_Z^\gamma(Z) = -(\lambda\sqrt{Z} - (\alpha - 1/2)/\sqrt{Z}) \quad (23)$$

Mimicking the Gaussian situation from Section 2 it is natural to measure distance to the gamma by comparing γ -scores with those in (23).

Definition 1 (Standardized gamma Fisher information). *The standardized $\gamma(\alpha, \lambda)$ -Fisher information of a positive random variable X with finite mean and pdf p is:*

$$J_{\text{st}, \gamma(\alpha, \lambda)}(X) = \frac{1}{\lambda} E \left[\left(\rho_X^\gamma(X) + \lambda\sqrt{X} - \frac{\alpha - 1/2}{\sqrt{X}} \right)^2 \right]. \quad (24)$$

Standardized gamma Fisher information is not location invariant (we need the input to be positive) but behaves nicely under scaling (under the assumption that $E[X] = \alpha/\lambda$):

$$J_{\text{st}, \gamma(\alpha, \lambda)}(aX) = J_{\text{st}, \gamma(\alpha, a\lambda)}(X) = \frac{1}{a} J_{\text{st}, \gamma(\alpha, \lambda)}(X) + \alpha \frac{(a - 1)^2}{a}. \quad (25)$$

Note that (by straightforward integration by parts starting from (22))

$$\sqrt{X} \rho_X^\gamma(X) = X \rho_X(X) + \frac{1}{2} \quad (26)$$

with $\rho_X(x) = (\log p_X(x))'$ the usual score of X (here we abuse notations slightly w.r.t. the definitions from Section 2). Hence we can rewrite (24) as

$$J_{\text{st}, \gamma(\alpha, \lambda)}(X) = \frac{1}{\lambda} E \left[X (\rho_X(X) + \lambda - (\alpha - 1)/X)^2 \right] \quad (27)$$

which is precisely the relative Fisher information advocated by [5]. Aiming at a Cramer-Rao inequality one might wish to expand the square in (24) in order to identify the correct gamma-Fisher information, but it is easy to realize that this will not yield good results. Following [1] we rather propose to introduce

$$I_{\gamma(\alpha, \lambda)}^r(X) = \frac{1}{\lambda} E \left[X \left(\rho_X(X) + \lambda(1 + r) - \frac{(\alpha - 1)}{X} \right)^2 \right]. \quad (28)$$

which we call a r -corrected gamma Fisher information. Clearly $I_{\gamma(\alpha, \lambda)}^r(Z) = \alpha r^2$ for all $r \geq 0$ and all $\lambda > 0$ if $Z \sim \gamma_{\alpha, \lambda}$ (recall that $\rho_Z(Z) = (\alpha - 1)/Z - \lambda$ in this case) and simple computations show that

$$J_{\text{st}, \gamma(\alpha, \lambda)}(X) = I_{\gamma(\alpha, \lambda)}^r(X) - \alpha r^2 \geq 0 \quad (29)$$

(we stress the important fact that this decomposition holds solely under a first moment assumption on X , see also [1, Remark 13]).

The relative entropy with respect to the gamma distribution is defined exactly as in the Gaussian case (recall (2)):

$$D(X || \gamma(\alpha, \lambda)) = \int_0^{+\infty} p_X(u) \log(p_X(u)/\gamma_{\alpha, \lambda}(u)) du \quad (30)$$

with X a random variable with density p_X on the positive real line. Note how gamma relative entropy does not behave as Gaussian relative entropy under scaling:

$$D(aX || \gamma(\alpha, \lambda)) = D(X || \gamma(\alpha, a\lambda)) \quad (31)$$

for all $a > 0$. There exists a De Bruijn identity specifically for (30), first identified by [4, 5] in the context of probability semigroup theory and Γ -calculus. We state a rescaling of the identity in its most general form as due to [1].

Theorem 2 (Gamma De Bruijn identity). *Let $\alpha \geq 1/2$ and suppose that X is a random variable with finite $\alpha + 4$ moments. Then*

$$\frac{d}{dr} D(X_r || \gamma(\alpha, \lambda/(1+r))) = \frac{1}{r} J_{st, \gamma(\alpha, \lambda)}(X_r) - \alpha \frac{r}{1+r} \quad (32)$$

where

$$X_r = \gamma(\alpha - \frac{1}{2}, \lambda) + \left(\sqrt{rX} + \frac{N}{\sqrt{2\lambda}} \right)^2, \quad (33)$$

with $\gamma(\alpha - \frac{1}{2}, \lambda)$ an independent gamma distributed random variable with parameters $\alpha - 1/2, \lambda$ and N as before an independent standard Gaussian random variable. The integrated version is

$$D(X || \gamma(\alpha, \lambda)) = \int_0^\infty \left(\frac{1}{r} J_{st, \gamma(\alpha, \lambda)}(X_r) - \alpha \frac{r}{1+r} \right) dr. \quad (34)$$

4 A quadratic gamma channel

Equations (32) and (33) lead us to introducing the nonlinear gamma channel (with all notations as in Theorem 2)

$$X \mapsto X_r (:= X_{r, \alpha, \lambda}) = \gamma(\alpha - 1/2, \lambda) + \left(\sqrt{rX} + \frac{N}{\sqrt{2\lambda}} \right)^2 \quad (35)$$

for $r > 0$. We also introduce the notation

$$Y_r = \sqrt{rX} + \frac{N}{\sqrt{2\lambda}} \quad (36)$$

Conditionally on X , the random variable X_r is the independent sum of a gamma and a non-central chi-squared random variable. This is the main difference between our channel (35) and classical “dual” channels wherein the distribution of the output, conditionally on the signal, remains within the same family of distributions as the noise (such as for instance in Gaussian channels studied in Section 2 or Poisson channels [13]).

Exploiting the moment generating function of X_r we obtain the following description of the channel.

Proposition 3. • *If X has moment generating function $M_X(\cdot)$ on $(0, a)$ then the moment generating function of X_r is*

$$M_r(t) = \left(1 - \frac{t}{\lambda} \right)^{-\alpha} M_X \left(\frac{rt}{1 - \frac{t}{\lambda}} \right) \quad (37)$$

on $(0, \lambda/(\lambda r/a + 1))$.

- *In particular if $E[X] = \alpha/\lambda$ then*

$$E[X_r] = \frac{\alpha}{\lambda} + rE[X] = \frac{\alpha}{\lambda}(1+r).$$

- *Let $\lambda_r \leq \lambda$. The output X_r is itself gamma distributed with parameters (α, λ_r) if and only if the input is gamma distributed with parameters $(\alpha, \frac{1}{r}(\frac{1}{\lambda_r} - \frac{1}{\lambda}))$.*

Proof. Identity (37) follows by independence as well as the fact that, conditionally on X , the random variable $\left(\sqrt{2\lambda r X} + N\right)^2$ is noncentral chi square distributed with non-centrality parameter $\sqrt{2\lambda r X}$. Hence

$$\begin{aligned} E[e^{tX_r}] &= \left(1 - \frac{t}{\lambda}\right)^{-(\alpha-1/2)} E\left[e^{t\left(\sqrt{rX} + \frac{N}{\sqrt{2\lambda}}\right)^2}\right] \\ &= \left(1 - \frac{t}{\lambda}\right)^{-(\alpha-1/2)} \frac{E\left[e^{\frac{rt}{1-t/\lambda}X}\right]}{(1-t/\lambda)^{1/2}}, \end{aligned}$$

which is defined as long as $t \leq \lambda$ and $rt/(1-t/\lambda) \leq a$. To see the next claim it suffices to notice that if $M_r(t) = (1-t/\lambda_r)^{-\alpha}$ then necessarily

$$M_X(t) = \left(1 - \frac{t}{r} \left(\frac{1}{\lambda_r} - \frac{1}{\lambda}\right)\right)^{-\alpha}$$

for t sufficiently small. □

Remark 4. An equivalent way to express the second point in Proposition 3: if X is gamma distributed with parameters α, λ_1 then X_r is gamma distributed with parameters (α, λ_r) where $\lambda_r = (\frac{r}{\lambda_1} + \frac{1}{\lambda})^{-1}$ for all $r > 0$.

5 Relative entropy and estimation theory

Proposition 5. Let $\alpha \geq 1/2$ and suppose that X is positive with finite mean. Define X_r, Y_r as in (35), (36) and introduce the ratio

$$\mathcal{V}_r(X) = \frac{Y_r}{\sqrt{X_r}}. \quad (38)$$

Then

$$\rho_r^\gamma(X_r) + \lambda\sqrt{X_r} - (\alpha - 1/2)/\sqrt{X_r} = E\left[\lambda\sqrt{rX} \mathcal{V}_r(X) \mid X_r\right] \quad (39)$$

and

$$J_{st,\gamma(\alpha,\lambda)}(X_r) = \lambda E\left[E\left[\sqrt{rX} \mathcal{V}_r(X) \mid X_r\right]^2\right]. \quad (40)$$

Remark 6. Note how in particular if $\alpha = 1/2$ then (38) reduces to $\text{sign}(Y_r)$, the sign of $\sqrt{rX} + N/\sqrt{2\lambda}$. This quantity plays a central role in Stein type representations for gamma specific Fisher information as obtained in [1, Proposition 23].

Remark 7. An immediate consequence of (40), Jensen's inequality for conditional expectations and the fact that $|\mathcal{V}_r(X)| \leq 1$ is the inequality

$$J_{st,\gamma(\alpha,\lambda)}(X_r) \leq \lambda r E[X] \quad (41)$$

for all $\lambda, r \geq 0$ and all $\alpha \geq 1/2$.

Proof. Identity (40) follows immediately from (39) and (24). To see (39) note how for all smooth test functions

$$\begin{aligned} &E\left[\left(\rho_r^\gamma(X_r) + \lambda\sqrt{X_r} - (\alpha - 1/2)/\sqrt{X_r}\right) \sqrt{X_r}\phi(X_r)\right] \\ &= E\left[\rho_r^\gamma(X_r)\sqrt{X_r}\phi(X_r)\right] + \lambda E[X_r\phi(X_r)] - E[(\alpha - 1/2)\phi(X_r)] \\ &= -E\left[\sqrt{X_r}\left(\frac{1}{2\sqrt{X_r}}\phi(X_r) + \sqrt{X_r}\phi'(X_r)\right)\right] + \lambda E[X_r\phi(X_r)] - E[(\alpha - 1/2)\phi(X_r)] \\ &= -\alpha E[\phi(X_r)] - E[X_r\phi'(X_r)] + \lambda E[X_r\phi(X_r)]. \end{aligned} \quad (42)$$

Expanding (33) we can rewrite the third summand as

$$\lambda E[X_r \phi(X_r)] = E \left[\lambda \gamma(\alpha - \frac{1}{2}, \lambda) \phi(X_r) \right] + E[\lambda r X \phi(X_r)] + \sqrt{2\lambda} E \left[\sqrt{rX} N \phi(X_r) \right] + \frac{1}{2} E[N^2 \phi(X_r)].$$

Applying (20) to the function $\gamma \mapsto \phi(\gamma + Y_r^2)$ we get

$$E \left[\lambda \gamma(\alpha - \frac{1}{2}, \lambda) \phi(X_r) \right] = (\alpha - \frac{1}{2}) E[\phi(X_r)] + E \left[\gamma(\alpha - \frac{1}{2}, \lambda) \phi'(X_r) \right]. \quad (43)$$

Applying (4) to the function $n \mapsto n \phi(\gamma(\alpha - \frac{1}{2}, \lambda) + (\sqrt{rX} + n/\sqrt{2\lambda})^2)$ we get

$$\frac{1}{2} E[N^2 \phi(X_r)] = \frac{1}{2} E[\phi(X_r)] + E \left[N \phi'(X_r) \frac{Y_r}{\sqrt{2\lambda}} \right]. \quad (44)$$

Applying (4) to $n \mapsto \sqrt{rX} \phi(\gamma(\alpha - \frac{1}{2}, \lambda) + (\sqrt{rX} + n/\sqrt{2\lambda})^2)$ we get

$$\sqrt{2\lambda} E \left[\sqrt{rX} N \phi(X_r) \right] = E \left[\sqrt{rX} \phi'(X_r) 2Y_r \right]. \quad (45)$$

Resuming from (42) we compute

$$\begin{aligned} & E \left[\left(\rho_r^\gamma(X_r) + \lambda \sqrt{X_r} - (\alpha - 1/2)/\sqrt{X_r} \right) \sqrt{X_r} \phi(X_r) \right] \\ &= E \left[\left\{ -\alpha + (\alpha - \frac{1}{2}) + \frac{1}{2} + \lambda r X \right\} \phi(X_r) \right] + E \left[\left\{ -X_r + \gamma(\alpha - \frac{1}{2}, \lambda) + \sqrt{rX} 2Y_r + N \frac{Y_r}{\sqrt{2\lambda}} \right\} \phi'(X_r) \right] \\ &= E[\lambda r X \phi(X_r)] + E \left[\sqrt{rX} Y_r \phi'(X_r) \right] \\ &= \lambda E \left[\sqrt{rX} \left(\sqrt{rX} + \frac{N}{\sqrt{2\lambda}} \right) \phi(X_r) \right], \end{aligned} \quad (46)$$

the last identity being a consequence of (45). By a standard density argument (identity (46) is valid for all smooth functions with compact support) we can then deduce the representation

$$\rho_r^\gamma(X_r) + \lambda \sqrt{X_r} - (\alpha - 1/2)/\sqrt{X_r} = \frac{\lambda E \left[\sqrt{rX} \left(\sqrt{rX} + \frac{N}{\sqrt{2\lambda}} \right) | X_r \right]}{\sqrt{X_r}} = \lambda E \left[\sqrt{rX} \mathcal{V}_r(X) | X_r \right] \quad (47)$$

which leads to (39). \square

Combining (40) with the gamma-specific De Bruijn identity (32) we immediately obtain that if X is a positive random variable with finite $\alpha + 4$ moment then

$$\frac{d}{dr} D(X_r || \gamma(\alpha, \lambda/(1+r))) = \frac{\lambda}{r} E \left[E \left[\sqrt{rX} \mathcal{V}_r(X) | X_r \right]^2 \right] - \alpha \frac{r}{r+1} \quad (48)$$

for all $r > 0$.

Example 8. Suppose that the input signal X is gamma distributed with parameters (α, λ) so that X_r follows a gamma law with parameters $(\alpha, \lambda/(1+r))$ for each $r > 0$ (recall Remark 4). Then, thanks to (47), we have

$$E \left[\sqrt{rX} \mathcal{V}_r(X) | X_r \right] = \frac{r}{1+r} \sqrt{X_r} \quad (49)$$

so that $J_{\text{st}, \gamma(\alpha, \lambda)}(X_r) = \frac{\alpha r^2}{r+1}$ and $\frac{d}{dr} D(X_r || \gamma(\alpha, \lambda/(1+r))) = 0$, as expected.

6 Mutual information and estimation theory

We start by restating identity (15) (which actually holds true for any channels) in the present gamma-target context. Let $r > 0$, $\tau(r) = r/(r+1)$ and introduce the random variables

$$\tilde{X}_{\tau(r)}(x) = (1 - \tau(r))\gamma(\alpha - \frac{1}{2}, \lambda) + (\sqrt{\tau(r)}x + \frac{\sqrt{1 - \tau(r)}}{\sqrt{2\lambda}}Z)^2, \quad (50)$$

$$X_r(x) = \gamma(\alpha, 1/2) + \left(\sqrt{rx} + \frac{N}{\sqrt{2\lambda}}\right)^2 \quad (51)$$

for $x \in [0, +\infty)$. We also write $X_r = X_r(X)$ and $\tilde{X}_{\tau(r)} = \tilde{X}_{\tau(r)}(X)$. Then $I(X, X_r) = I(X, \tilde{X}_{\tau(r)})$ and

$$I(X, \tilde{X}_{\tau(r)}) = E_X \left[D(\tilde{X}_{\tau(r)}(X) \parallel \gamma(\alpha, \lambda)) \right] - D(\tilde{X}_{\tau(r)} \parallel \gamma(\alpha, \lambda)), \quad (52)$$

where $E_X[\cdot]$ denotes an expectation taken with respect to X as in (14). Similarly as in Section 2 we also deduce from the gamma-De Bruijn identity (see Theorem 14 of [1]) as well as the chain rule for differentiation:

Lemma 9. *Let $\alpha \geq 1/2$, $\lambda > 0$ and $r > 0$. If X is almost surely positive with finite $\alpha + 4$ moments and mean $E[X] = \alpha/\lambda$ then*

$$\frac{d}{dr} \left(I(X, X_r) \right) = \frac{1}{r(r+1)} \left(E_X \left[J_{st, \gamma(\alpha, \lambda)}(\tilde{X}_{\tau(r)}(X)) \right] - J_{st, \gamma(\alpha, \lambda)}(\tilde{X}_{\tau(r)}) \right), \quad (53)$$

We are now in a position to obtain the gamma counterpart to the GSV identity (13). However, as already pointed out, the problem with the quadratic gamma channel is that it is more difficult to compute directly $J_{st, \gamma(\alpha, \lambda)}(\tilde{X}_{\tau(r)}(x))$ because $\tilde{X}_{\tau(r)}(x)$ is not a gamma random variable but rather a non-central gamma whose explicit density is complicated to manipulate.

Proposition 10. *Let $\alpha \geq 1/2$, $\lambda > 0$, $r > 0$ and X be a positive random variable with finite $\alpha + 4$ moment and mean equal to α/λ . Then*

$$\frac{d}{dr} \left(I(X, X_r) \right) = \lambda \left(E_X \left[X E[E[\mathcal{V}_r(X) | X_r(X)]^2] \right] - E[E[\sqrt{X} \mathcal{V}_r(X) | X_r]^2] \right). \quad (54)$$

Proof. First of all, applying Lemma 9, we have:

$$\frac{d}{dr} \left(I(X, X_r) \right) = \frac{1}{r(r+1)} \left(\int p_X(x) J_{st, \gamma(\alpha, \lambda)}(\tilde{X}_{\tau(r)}(x)) dx - J_{st, \gamma(\alpha, \lambda)}(\tilde{X}_{\tau(r)}) \right), \quad (55)$$

Applying a slight extension of (25) to $\tilde{X}_{\tau(r)} = \frac{1}{r+1}X_r$ we deduce

$$J_{st, \gamma(\alpha, \lambda)}(\tilde{X}_{\tau(r)}) = (1+r)J_{st, \gamma(\alpha, \lambda)}(X_r) - \alpha r^2. \quad (56)$$

Applying Proposition 5 then leads to

$$J_{st, \gamma(\alpha, \lambda)}(\tilde{X}_{\tau(r)}) = (1+r)\lambda E[E[\sqrt{rX} \mathcal{V}_r(X) | X_r]^2] - \alpha r^2. \quad (57)$$

Now, Proposition 5 is true irrespectively of the distribution of the input, so that we also have (for each fixed x)

$$J_{st, \gamma(\alpha\lambda)}(X_r(x)) = \lambda E[E[\sqrt{rx} \mathcal{V}_r(x) | X_r(x)]^2].$$

Furthermore, we have

$$J_{st, \gamma(\alpha, \lambda)}(\tilde{X}_{\tau(r)}(x)) = (1+r)J_{st, \gamma(\alpha, \lambda)}(X_r(x)) - 2r^2\lambda x + \frac{\lambda r^2}{(1+r)} \left(\frac{\alpha}{\lambda} + rx \right), \quad (58)$$

which leads to

$$J_{st,\gamma(\alpha,\lambda)}(\tilde{X}_{\tau(r)}(x)) = (1+r)\lambda E[E[\sqrt{rx}\mathcal{V}_r(x) | X_r(x)]^2] - 2r^2\lambda x + \frac{\lambda r^2}{(1+r)}\left(\frac{\alpha}{\lambda} + rx\right). \quad (59)$$

Integrating the previous expression with respect to the density of X together with the fact that $E[X] = \frac{\alpha}{\lambda}$, we obtain

$$\int_{\mathbb{R}_+^*} J_{st,\gamma(\alpha,\lambda)}(\tilde{X}_{\tau(r)}(x)) p_X(x) dx = (1+r) \int_{\mathbb{R}_+^*} \lambda E[E[\sqrt{rx}\mathcal{V}_r(x) | X_r(x)]^2] p_X(x) dx \quad (60)$$

$$- 2r^2\lambda \int_{\mathbb{R}_+^*} x p_X(x) dx + \frac{\lambda r^2}{1+r} \left(\frac{\alpha}{\lambda} + r \int_{\mathbb{R}_+^*} x p_X(x) dx \right), \quad (61)$$

$$= (1+r) \int_{\mathbb{R}_+^*} \lambda E[E[\sqrt{rx}\mathcal{V}_r(x) | X_r(x)]^2] p_X(x) dx - 2r^2\alpha + \alpha r^2, \quad (62)$$

$$= (1+r) \int_{\mathbb{R}_+^*} \lambda E[E[\sqrt{rx}\mathcal{V}_r(x) | X_r(x)]^2] p_X(x) dx - \alpha r^2. \quad (63)$$

Combining (63) and (57) together with (55), we obtain the relation:

$$\frac{d}{dr} \left(I(X, X_r) \right) = \lambda \left(\int_{\mathbb{R}_+^*} x E[E[\mathcal{V}_r(x) | X_r(x)]^2] p_X(x) dx - E[E[\sqrt{X}\mathcal{V}_r(X) | X_r]^2] \right) \quad (64)$$

leading directly to the claim. \square

Remark 11. *It should be clear that the previous result holds true even if $E[X] \neq \alpha/\lambda$. The proof is similar by using the general relation,*

$$J_{st,\gamma(\alpha,\lambda)}(\tilde{X}_{\tau(r)}) = (1+r)J_{st,\gamma(\alpha,\lambda)}(X_r) - 2r^2\lambda E[X] + \frac{\lambda r^2}{(1+r)}\left(\frac{\alpha}{\lambda} + rE[X]\right), \quad (65)$$

instead of (56).

6.1 An upper bound

An immediate consequence of (54) and the fact that $|\mathcal{V}_r(X)| \leq 1$ is the upper bound

$$\frac{d}{dr} \left(I(X, X_r) \right) \leq \lambda \left(E[X] - E[E[\sqrt{X}\mathcal{V}_r(X) | X_r]^2] \right). \quad (66)$$

Note that (66) is very close to the Gaussian GSV identity (13). In particular when $X \sim \gamma_{\alpha,\lambda}$, using (49) and (66), we have:

$$\frac{d}{dr} \left(I(X, X_r) \right) \leq \frac{\alpha}{1+r}, \quad (67)$$

which leads to the fine bound:

$$I(X, X_r) \leq \alpha \log(1+r). \quad (68)$$

The previous bound should be compared with the corresponding formula (11) of [10] which is satisfied by the mutual information of the additive Gaussian channel with Gaussian input. When $X \sim \gamma_{\alpha,\nu}$, we have the bound:

$$I(X, X_r) \leq \alpha \log \left(1 + \frac{\lambda r}{\nu} \right). \quad (69)$$

6.2 A lower bound for $\alpha = \frac{1}{2}$

We set $\alpha = \frac{1}{2}$. Assume that the input is gamma distributed with parameters $(1/2, \lambda)$. By definition, the mutual information between X and X_r is equal to:

$$I(X, X_r) = \int p_{X_r|X=x}(u, x) p_X(x) \log \left(\frac{p_{X_r|X=x}(u, x)}{p_{X_r}(u)} \right) du dx$$

Let us compute explicitly the ratio between $p_{X_r|X=x}(u, x)$ and $p_{X_r}(u)$ in order to provide a simple lower bound for the logarithm term in the previous expression. We have:

$$\begin{aligned} \frac{p_{X_r|X=x}(u, x)}{p_{X_r}(u)} &= \Gamma(\alpha) \frac{\lambda e^{-\lambda u} e^{-\lambda r x} \left(\frac{u}{rx} \right)^{\frac{\alpha-1}{2}} I_{\alpha-1}(2\lambda\sqrt{uxr})}{u^{\alpha-1} e^{-\frac{\lambda u}{r+1}} \left(\frac{\lambda}{r+1} \right)^\alpha}, \\ &= \Gamma(\alpha) (1+r)^\alpha \frac{1}{\lambda^{\alpha-1}} e^{-\frac{\lambda r u}{r+1}} e^{-\lambda r x} \frac{I_{\alpha-1}(2\lambda\sqrt{uxr})}{(urx)^{\frac{\alpha-1}{2}}}. \end{aligned}$$

Moreover, since $I_{-1/2}(z) = \sqrt{2/\pi} \cosh(z)/\sqrt{z}$, we obtain:

$$\frac{p_{X_r|X=x}(u, x)}{p_{X_r}(u)} = \sqrt{\pi} \sqrt{(1+r)} \frac{1}{\lambda^{-\frac{1}{2}}} e^{-\frac{\lambda r u}{r+1}} e^{-\lambda r x} \frac{\cosh(2\lambda\sqrt{uxr})}{\sqrt{\pi\lambda}}, \quad (70)$$

$$= \sqrt{(1+r)} e^{-\frac{\lambda r u}{r+1}} e^{-\lambda r x} \cosh(2\lambda\sqrt{uxr}), \quad (71)$$

$$\geq \frac{1}{2} \sqrt{(1+r)} e^{-\frac{\lambda r u}{r+1}} e^{-\lambda r x} e^{2\lambda\sqrt{uxr}}. \quad (72)$$

Using the monotonicity of the logarithm, we obtain:

$$\log \left(\frac{p_{X_r|X=x}(u, x)}{p_{X_r}(u)} \right) \geq \log \left(\frac{1}{2} \sqrt{(1+r)} e^{-\frac{\lambda r u}{r+1}} e^{-\lambda r x} e^{2\lambda\sqrt{uxr}} \right). \quad (73)$$

This inequality implies the following on the mutual information between X and X_r :

$$I(X, X_r) \geq \frac{1}{2} \log(1+r) - \log(2) - \frac{\lambda r}{r+1} E[X_r] - r\lambda E[X] + 2\lambda\sqrt{r} E[\sqrt{X_r}\sqrt{X}], \quad (74)$$

$$\geq \frac{1}{2} \log(1+r) - \log(2) - r\alpha - r\alpha + 2\lambda\sqrt{r} E[\sqrt{r}X + \frac{Z\sqrt{X}}{\sqrt{2\lambda}}], \quad (75)$$

$$\geq \frac{1}{2} \log(1+r) - \log(2) - 2r\alpha + 2\alpha r = \frac{1}{2} \log(1+r) - \log(2), \quad (76)$$

where we have used the fact $|x| \geq x$, X and Z are independent and $E[Z] = 0$. This lower bound combined with the bound (68) implies that:

$$\lim_{r \rightarrow +\infty} \frac{I(X, X_r)}{\frac{1}{2} \log(1+r)} = 1 \quad (77)$$

Remark 12. • Thus, for $\alpha = 1/2$ and for a gamma- $(1/2, \lambda)$ distributed input, the mutual information between X and the output X_r exhibits the same asymptotic for large values of the channel quality parameter r as the mutual information between the additive Gaussian channel and a Gaussian input.

- It would be nice to know if such an asymptotic is still true for $\alpha > 1/2$ and a gamma- (α, λ) distributed input. More generally we ask the question: for which input distribution do we have the same type of asymptotic as in (77) for the mutual information? Such questions are related to the concept of “MMSE dimension”, see [31].

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